Math 255B Lecture 20 Notes

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1 Development of Spectral Measures

1.1 Nevanlinna-Herglotz functions

Last time, we were proving the following theorem from complex analysis.

Theorem 1.1 (Nevanlinna, Herglotz,...). *Let f be a holomorphic function in $\operatorname{Im} z > 0$ with $\operatorname{Im} f \geq 0$ and $|f(z)| \leq \frac{c}{\operatorname{Im} z}$. Then there is a uniform bound*

$$\int \operatorname{Im} f(x+iy)\, dx \leq C\pi \qquad \forall y > 0,$$

and there exists a positive bounded measure μ on $\mathbb{R}$ such that

$$\frac{1}{\pi} \int \varphi(x) \operatorname{Im} f(x+iy)\, dx \xrightarrow{y \to 0^+} \int \varphi\, d\mu \qquad \forall \varphi \in C_B := (C \cap L^\infty)(\mathbb{R}).$$

We have

$$f(z) = \int \frac{1}{\xi - z} d\mu(\xi), \qquad \operatorname{Im} z > 0,$$

$$\int d\mu(\xi) = \lim_{y \to +\infty} y \operatorname{Im} f(iy) = \lim_{z \to \infty} (-z f(z)),$$

where $z \to \infty$ with $\arg(z)$ bounded away from $0, \pi$.

Conversely, if $\mu \geq 0$ is a bounded measure on $\mathbb{R}$ and f is defined by $f(z) = \int \frac{1}{\xi - z} d\mu(\xi)$, then both the weak convergence and the limit condition hold.

Last time, we showed that $f(z) = \int \frac{1}{\xi - z} d\mu(\xi)$, which implies that

$$\int d\mu(\xi) = \lim_{y \to \infty} y \operatorname{Im} f(iy) = \lim_{z \to \infty} (-z f(z)).$$

Proof. We claim that $\frac{1}{\pi}\operatorname{Im} f(x+iy)\,dx \to d\mu$ weak* as $y\to 0^+$. If $\varphi \in C_B(\mathbb{R})$,

$$\begin{aligned}\frac{1}{\pi}\operatorname{Im} f(x+iy)\varphi(x)\,dx &= \frac{1}{\pi}\iint \varphi(x)\frac{y}{|\xi-x-iy|^2}d\mu(\xi)\,dx \\ &= \int\left(\frac{y}{\pi}\int\frac{\varphi(x)}{|\xi-x-iy|^2}\,dx\right)\,d\mu(\xi)\end{aligned}$$

The part inside the parentheses is $\frac{1}{\pi}\int\frac{\varphi(\xi+ty)}{1+t^2}\,dt \to \varphi(\xi)$ by cominated convergence as $y\to 0^+$. The convergence is dominated by $\|\varphi\|_{L^\infty}\in L^1(d\mu)$, so by another application of dominated convergence, we get

$$\xrightarrow{y\to 0^+} \int \varphi(\xi)\,d\mu(\xi). \qquad \square$$

1.2 Construction of spectral measures via Nevanlinna-Herglotz functions

Apply the theorem to $f(z)=\langle R(z)u,u\rangle$, where $u\in H$ and $R(z)=(A-z)^{-1}$ is the resolvent of a self-adjoint operator A. We can write

$$\langle R(z)u,u\rangle = \int\frac{1}{\xi-z}\,d\mu_u(\xi),$$

where the total measure is

$$\int d\mu_u = \lim_{y\to+\infty} y\underbrace{\operatorname{Im}\langle R(iy)u,u)\rangle}_{y\|R(iy)u\|^2} \le \|u\|^2.$$

We have, for $\varphi\in C_B$,

$$\begin{aligned}\int\varphi(x)\,d\mu_u(x) &= \lim_{y\to 0^+}\frac{1}{\pi}\int\operatorname{Im}\langle R(x+iy)u,u\rangle\,\varphi(x)\,dx \\ &= \lim_{y\to 0^+}\frac{1}{2\pi i}\varphi(x)[\langle R(x+iy)u,u\rangle - \langle R(x-iy)u,u\rangle]\,dx\end{aligned}$$

So the measure is given by the jump of the resolvent over the real line.

Remark 1.1. We have that $\operatorname{supp}(\mu_n)\subseteq \operatorname{Spec}(A)$ for all u.

Define the complex measures $\mu_{u,v}$ for $u,v\in H$ by polarization:

$$d\mu_{u,v} = \frac{1}{4}(d\mu_{u+v}+id\mu_{u+iv}-d\mu_{u-v}-id\mu_{u-iv}).$$

Then we have

$$\langle R(z)u,v\rangle = \int\frac{1}{\xi-z}\,d\mu_{u,v}(\xi),$$

$$\int \varphi \, d\mu_{u,v} = \lim_{y \to 0^+} \frac{1}{2\pi i} \int \varphi(x) \left\langle (R(x+iy) - R(x-iy))u, v \right\rangle \, dx.$$

The total mass of $\mu_{u,v}$ is

$$\int d\mu_{u,v} \leq \frac{1}{4}(\|u+v\|^2 + \|u+iv\|^2 + \|u-v\|^2 + \|u-iv\|$$

By homogeneity, we may assume $\|u\| = \|v\| = 1$.

$$= \frac{1}{4}(2(\|u\|^2 + \|v\|^2)2) = 2.$$

So by homogeneity, $\|\mu_{u,v}\| \leq 2\|u\| \cdot \|v\|$.

For fixed $\varphi \in C_B$, we have

$$\left| \int \varphi \, d\mu_{u,v} \right| \leq 2\|\varphi\|_{L^\infty} \|u\| \|v\|,$$

so the map $(u,v) \mapsto \int \varphi \, d\mu_{u,v}$ is a bounded, sesquilinear form on $H \times H$. By the Riesz representation theorem, there is a unique operator $\varphi(A) \in \mathcal{L}(H,H)$ such that $\langle \varphi(A)u, v \rangle = \int \varphi \, d\mu_{u,v}$. We also get that $\|\varphi(A)\| \leq 2\|\varphi\|_{L^\infty}$. In other words,

$$\varphi(A) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int \varphi(\lambda)(R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)) \, d\lambda$$

(where the limit exists in the weak operator topology).

Remark 1.2. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a Hermitian matrix. Then the above relation holds: it suffices to check this formula when $n = 1$, where A is multiplication by t for $t \in \mathbb{R}$. In this case, this is

$$\varphi(t) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int \varphi(\lambda) \left(\frac{1}{t - \lambda - i\varepsilon} - \frac{1}{t - \lambda + i\varepsilon} \right) d\lambda.$$

This is equivalent to

$$\delta(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \left(\frac{1}{t - i\varepsilon} - \frac{1}{t + i\varepsilon} \right),$$

which is called **Plemelj's formula**. The proof is (if we take $t = 0$):

$$\varphi(0) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int \varphi(t) \frac{2i\varepsilon}{t^2 + \varepsilon^2} = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int \frac{\varepsilon \varphi(t)}{\varepsilon^2 + t^2}.$$

One sometimes writes

$$\delta(A - \lambda) = \frac{1}{2\pi i}(R(\lambda + i0) - R(\lambda - i0)),$$

$$\varphi(A) = \int \varphi(\lambda) \delta(A - \lambda) \, d\lambda.$$

Definition 1.1. The measures $\mu_{u,v}$ are called the **spectral measures** of A.

We have defined a map of algebras $C_B \to \mathcal{L}(H,H)$ sending $\varphi \mapsto \varphi(A)$. This map has nice algebraic properties that we will study. This will lead us to the development of the spectral theorem.